Location of Zeros Of Polar Derivative of Polynomials With Real Coefficients

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Abstract: In this paper we obtain the size of the disc in which the zeros of polar derivatives of polynomial of degree n with real coefficients with respect to a real α lie.

Keywords: zeros, polar derivatives, polynomials, real α.

INTRODUCTION:

To estimate the zeros of a polynomial is a long standing problem. It is an interesting area of research for many engineers as well as mathematicians and many results on the topic are available in the literature.

If $P(z)=\sum_{i=0}^{n} a_i z^i$, be a polynomial of degree n then Polar Derivative of the polynomial P(z) with respect to α, where α can be real or complex number, is defined as

$$D_\alpha P(z) = n P(z) + (\alpha - z) P'(z).$$

It is a polynomial of degree up to n-1. The polynomial $D_\alpha P(z)$ generalizes the ordinary derivative, in the sense that $\lim_{n\to\infty} D_\alpha P(z)/\alpha = P'(z)$.

This paper we prove the following results.

Theorem (1): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that

$$a_n \le a_{n-1} \le \ldots \le a_0 < 0$$

and $ia_i \le (i-1)a_{i-1}$ i= 0, 1, 2, . . .n-2.

Then the polar derivative of P(z) with respect to a real $\alpha \ne -a_{n-1}/na_n$ has (n-1) roots and they lie in

$$|z| \le 1.$$

Theorem (2): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that

$$a_n \le a_{n-1} \le \ldots \le a_0$$

and $ia_i \le (i-1)a_{i-1}$ i= 0, 1, 2, . . .n-2.

Then the polar derivative of P(z) with respect to $\alpha \ne -a_{n-1}/na_n$ has (n-1) roots and they lie in

$$|z| \le |a_{n-1} + \alpha n a_n|^{-1}\{-(a_{n-1} + \alpha n a_n) + na_0 + \alpha a_1 + |\, na_0 + \alpha a_1 \,|\}.$$

Theorem (3): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that

$a_m \le a_{m-1} \le \ldots \le a_0$ where m= 0, 1, 2……n

and $ia_i \le (i-1)a_{i-1}$ i= 0, 1, 2, . . .m-2.

Then the polar derivative of P(z) with respect to α such that

$\alpha = -a_{n-1}/ na_n = -2a_{n-2}/(n-1)a_{n-1}=\ldots=-(n-m-1)a_{m+1}/(m+2)a_{m+2}$ has exactly m roots and they lie in

$|z| \le |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{-(n-m)a_m -\alpha(m+1)a_{m+1} + na_0$

$+\alpha na_1 + | na_0 + \alpha na_1 |\}$.

PROOF OF THEOREM 1:

Let $P(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n$ be a polynomial of degree n.

Then the polar derivative of P(z) is given by $D_\alpha P(z) = n P(z) + (\alpha-z) P'(z)$. Then

$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2] z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$

$+ [(n-m+1)a_{m-1} + \alpha ma_m]z^{m-1} + [(n-m)a_m +\alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} +\alpha(m+2)a_{m+2}]z^{m+1} + \ldots +$

$[2a_{n-2} +\alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha na_n]z^{n-1}$.

Now consider the polynomial Q(z) = (1-z) $D_\alpha P(z)$ so that

$Q(z) = -[a_{n-1} + \alpha na_n]z^n + [a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots$

$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$

$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m]z^m$

$+ [(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$

$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$

$+ [na_0 + \alpha a_1]$.

Now if $|z|>1$ then $|z|^{i-n} < 1$ for i = 1, 2, 3, …..n-2

Further

$|Q(z)| \ge |a_{n-1} + \alpha na_n| |z|^n - \{|a_{n-1} + \alpha na_n - 2a_{n-2} - \alpha(n-1)a_{n-1}| |z|^{n-1}$

$+ \ldots + |(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}| |z|^{m+1}$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m || z|^m$

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$+|(n-m+1)a_{m-1} + \alpha m a_m – (n-m+2)a_{m-2} – \alpha(m-1)a_{m-1}|\ |z|^{m-1}$

$+|(n-2)a_2 + 3\alpha a_3 – (n-1)a_1 - 2\alpha a_2|\ |z|^2 + |(n-1)a_1 + 2\alpha a_2 – na_0 - \alpha a_1|\ |z|$

$+|na_0 + \alpha a_1|\}.$

$\geq |a_{n-1} + \alpha n a_n|\ |z|^n - |a_{n-1} + \alpha n a_n|^{-1}\{|a_{n-1} + \alpha n a_n - 2a_{n-2}$

$-\alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}|\ |z|^{-1} + \dots$

$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\ |z|^{-(n-m-2)}$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m||\ z|^{-(n-m-1)}$

$+|(n-m+1)a_{m-1} + \alpha m a_m – (n-m+2)a_{m-2} – \alpha(m-1)a_{m-1}|\ |z|^{-(n-m)} + \dots$

$+|(n-2)a_2 + 3\alpha a_3 – (n-1)a_1 - 2\alpha a_2|\ |z|^{-(n-3)} + |(n-1)a_1 + 2\alpha a_2$

$– na_0 - \alpha a_1|\ |z|^{-(n-2)} + |na_0 + \alpha a_1|\ |z|^{-(n-1)}\}].$

$\geq |a_{n-1} + \alpha n a_n|\ |z|^{n-1}[|z| - |a_{n-1} + \alpha n a_n|^{-1}\{|a_{n-1} + \alpha n a_n - 2a_{n-2}$

$-\alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}| + \dots$

$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|$

$+|(n-m+1)a_{m-1} + \alpha m a_m – (n-m+2)a_{m-2} – \alpha(m-1)a_{m-1}| + \dots$

$+|(n-2)a_2 + 3\alpha a_3 – (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 – na_0 - \alpha a_1| + |na_0 + \alpha a_1|\}].$

$\geq |a_{n-1} + \alpha n a_n|\ |z|^{n-1}[|z| - |a_{n-1} + \alpha n a_n|^{-1}\{-(a_{n-1} + \alpha n a_n) + 2a_{n-2}$

$+\alpha(n-1)a_{n-1} - 2a_{n-2} - \alpha(n-1)a_{n-1} + 3a_{n-3} + \alpha(n-2)a_{n-2} + \dots$

$-(n-m-1)a_{m+1} - \alpha(m+2)a_{m+2} + (n-m)a_m + \alpha(m+1)a_{m+1}$

$-(n-m)a_m - \alpha(m+1)a_{m+1} + (n-m+1)a_{m-1} + \alpha m a_m$

$-(n-m+1)a_{m-1} - \alpha m a_m + (n-m+2)a_{m-2} + \alpha(m-1)a_{m-1} + \dots$

$-(n-2)a_2 - 3\alpha a_3 + (n-1)a_1 + 2\alpha a_2 - (n-1)a_1 - 2\alpha a_2 + na_0 + \alpha a_1$

$+|na_0 + \alpha a_1|\}].$

$\geq |a_{n-1} + \alpha n a_n|\ |z|^{n-1}[|z| - 1].$

>0 if $|z| > 1$

This shows that if

$|z| > 1$ then $Q(z) > 0$.

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$|z| \le 1$

But those zeros of Q(z) whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of Q(z) as they lie in the circle defined by the above inequality and this completes the proof.

PROOF OF THEOREM 2:

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n$ be a polynomial of degree n.

Then the polar derivative of P(z) is given by $D_\alpha P(z) = n\,P(z) + (\alpha - z)\,P'(z)$. Then

$$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2]\,z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$$

$$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots + [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha n a_n]z^{n-1}.$$

Now consider the polynomial $Q(z) = (1-z)\,D_\alpha P(z)$ so that

$$Q(z) = -[a_{n-1} + \alpha n a_n]z^n + [a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots.$$

$$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$$

$$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m$$

$$+ [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$$

$$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$$

$$+ [na_0 + \alpha a_1].$$

Now if $|z|>1$ then $|z|^{i-n} < 1$ for i = 1, 2, 3, ...n-2

Further

$$|Q(z)| \ge |a_{n-1} + \alpha n a_n|\,|z|^n - \{|a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}|\,|z|^{n-1}$$

$$+ \ldots + |(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\,|z|^{m+1}$$

$$+ |(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|\,|z|^m$$

$$+ |(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\,|z|^{m-1}$$

$$+ |(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\,|z|^2 + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\,|z|$$

$$+ |na_0 + \alpha a_1|\}.$$

$\geq |a_{n-1} + \alpha na_n| \; |z|^{n-1}[|z| - |a_{n-1} + \alpha na_n|^{-1}\{|a_{n-1} + \alpha na_n - 2a_{n-2}$

$-\alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}| \; |z|^{-1} + \ldots$

$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}| \; |z|^{-(n-m-2)}$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m| \, |z|^{-(n-m-1)}$

$+|(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| \; |z|^{-(n-m)} + \ldots$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| \; |z|^{-(n-3)} + |(n-1)a_1 + 2\alpha a_2$

$- na_0 - \alpha a_1| \; |z|^{-(n-2)} + |na_0 + \alpha a_1| \; |z|^{-(n-1)} \}].$

$\geq |a_{n-1} + \alpha na_n| \; |z|^{n-1}[|z| - |a_{n-1} + \alpha na_n|^{-1}\{|a_{n-1} + \alpha na_n - 2a_{n-2}$

$-\alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}| + \ldots$

$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m|$

$+|(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| + \ldots$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1| + |na_0 + \alpha a_1| \}].$

$\geq |a_{n-1} + \alpha na_n| \; |z|^{n-1}[|z| - |a_{n-1} + \alpha na_n|^{-1}\{-(a_{n-1} + \alpha na_n) + 2a_{n-2}$

$+\alpha(n-1)a_{n-1} - 2a_{n-2} - \alpha(n-1)a_{n-1} + 3a_{n-3} + \alpha(n-2)a_{n-2} + \ldots$

$-(n-m-1)a_{m+1} - \alpha(m+2)a_{m+2} + (n-m)a_m + \alpha(m+1)a_{m+1}$

$-(n-m)a_m - \alpha(m+1)a_{m+1} + (n-m+1)a_{m-1} + \alpha ma_m$

$-(n-m+1)a_{m-1} - \alpha ma_m + (n-m+2)a_{m-2} + \alpha(m-1)a_{m-1} + \ldots$

$-(n-2)a_2 - 3\alpha a_3 + (n-1)a_1 + 2\alpha a_2 - (n-1)a_1 - 2\alpha a_2 + na_0 + \alpha a_1$

$+|na_0 + \alpha a_1|\}].$

$\geq |a_{n-1} + \alpha na_n| \; |z|^{n-1}[|z| - |a_{n-1} + \alpha na_n|^{-1}\{-(a_{n-1} + \alpha na_n)$

$+ na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}].$

>0 if $|z| > |a_{n-1} + \alpha na_n|^{-1}\{-(a_{n-1} + \alpha na_n) + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$

This shows that if

$|z| > |a_{n-1} + \alpha na_n|^{-1}\{-(a_{n-1} + \alpha na_n) + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$ then $Q(z) > 0$.

Hence all the zeros of Q(z) with $|z|>1$ lie in

$|z| \leq |a_{n-1} + \alpha n a_n|^{-1}\{-(a_{n-1} + \alpha n a_n) + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$.

But those zeros of Q(z) whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of Q(z) as they lie in the circle defined by the above inequality and this completes the proof.

PROOF OF THEOREM 3:

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n$ be a polynomial of degree n.

Then the polar derivative of P(z) is given by $D_\alpha P(z) = n\,P(z) + (\alpha - z)\,P'(z)$. Then

$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2]\,z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$

$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots + [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha n a_n]z^{n-1}$.

As $\alpha = -a_{n-1}/na_n = -2a_{n-2}/(n-1)a_{n-1} = \ldots.. = -(n-m-1)a_{m+1}/(m+2)a_{m+2}$

$D_\alpha P(z) = [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots$

$+ [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha n a_n]z^{n-1}$.

Now consider the polynomial $Q(z) = (1-z)\,D_\alpha P(z)$ so that

$Q(z) = -[(n-m)a_m + \alpha(m+1)a_{m+1}]z^{m+1} + [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1}$

$- \alpha m a_m]z^m + [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$

$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$

$+ [na_0 + \alpha a_1]$.

Now if $|z|>1$ then $|z|^{i-m} < 1$ for i = 1, 2, 3, …..m-2

Further,

$|Q(z)| \geq |(n-m)a_m + \alpha(m+1)a_{m+1}|\,|z|^{m+1} - \{|(n-m)a_m + \alpha(m+1)a_{m+1}$

$- (n-m+1)a_{m-1} - \alpha m a_m|\,|z|^m + |(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2}$

$- \alpha(m-1)a_{m-1}|\,|z|^{m-1} + \ldots$

$+ |(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\,|z|^2 + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\,|z|$

$+ |na_0 + \alpha a_1|$.

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m \, [|z|$

$-|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{|(n-m)a_m + \alpha(m+1)a_{m+1}$

$- (n-m+1)a_{m-1} - \alpha m a_m |+\ldots+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| \, |z|^{-(m-2)}$

$+|(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1 ||z|^{-(m-1)}+|na_0 + \alpha a_1||z|^{-m}\}]$.

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m \, [|z|$

$-|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{|(n-m)a_m + \alpha(m+1)a_{m+1}$

$- (n-m+1)a_{m-1} - \alpha m a_m |+\ldots +|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|$

$+|(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1 |+|na_0 + \alpha a_1|\}]$.

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m \, [|z|$

$-|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{-[(n-m)a_m + \alpha(m+1)a_{m+1}]$

$+(n-m+1)a_{m-1} + \alpha m a_m -\ldots -(n-2)a_2 - 3\alpha a_3 + (n-1)a_1 + 2\alpha a_2$

$-(n-1)a_1 - 2\alpha a_2 + na_0 + \alpha a_1 +|na_0 + \alpha a_1|\}]$.

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m \, [|z|$

$-|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{- [(n-m)a_m + \alpha(m+1)a_{m+1}]+na_0 + \alpha a_1$

$+|na_0 + \alpha a_1|\}]$.

>0 if $|z| > |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{-(n-m)a_m - \alpha(m+1)a_{m+1} +na_0 + \alpha a_1 +|na_0 + \alpha a_1|\}$.

This shows that if

$|z| > |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{-(n-m)a_m - \alpha(m+1)a_{m+1} +na_0 + \alpha a_1 +|na_0 + \alpha a_1|\}$.

then $Q(z) > 0$.

Hence all the zeros of Q(z) with $|z|>1$ lie in

$|z| > |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{-(n-m)a_m -\alpha(m+1)a_{m+1} +na_0 + \alpha a_1 +|na_0 + \alpha a_1|\}$.

But those zeros of Q(z) whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of Q(z) as they lie in the circle defined by the above inequality and this completes the proof.

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